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Some Economical Classes of Minimal Circular Generalised Neighbour Designs

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Neighbour designs are used to balance out the neighbour effects in the experiments where the effect of a treatment is influenced by the effects of treatment(s) applied to the neighbouring units. Minimal neighbour designs are always economical, therefore, preferred by the experimenters. In circular blocks, minimal neighbour designs can be constructed for *v* odd, where *v* is number of treatments. Minimal circular neighbour designs cannot be constructed for most of the cases when *v* is even. In such situations, generalized neighbour designs are used. In this article, some construction procedures are developed to obtain some classes of minimal circular generalized neighbour designs in blocks of (i) two different sizes, and (ii) three different sizes.

Key words: *Neighbour effects; Neighbour balanced designs; GNDs; GN_2-designs; GN_3-designs.*

Mathematics Subject Classification (2010): 05B05; 62K10; 62K05.

1. Introduction

Neighbour designs are used to balance out the neighbour effects in the experiments where effect of a treatment is influenced by the effects of treatment(s) applied to the neighbouring units. Neighbour effects are the major source of bias, therefore,Azais (1987) suggested that the bias due to neighbour effects is in some sense minimal if neighbour designs are used. A design in which each pair of non-identical treatments appears equally often in adjacent plots of the same circular block is called a neighbour design. A design in which each pair of non-identical treatments appears exactly once in adjacent plots of the same block is called minimal neighbour design. Minimal neighbour designs are always economical, therefore, preferred by the experimenters. In circular blocks, minimal neighbour designs can be constructed for odd v number of treatments. Rees (1967) used neighbour designs in virus research and constructed these designs for some cases of v odd. Hwang (1973), Azais *et al.* (1993), Iqbal *et al.* (2009),Akhtar *et al*. (2010), Ahmed and Akhtar (2011a) constructed neighbour designs for some specific cases. Minimal neighbour designs cannot be constructed for most of the cases when v is even. In such situations, generalised neighbour designs are used..A circular design in which most pairs of distinct treatments appear as neighbours once and remaining pairs appear twice is called minimal circular generalised neighbour design (CGND). Misra *et al*. (1991), Chaure and Misra (1996), Nutan (2007), Kedia and Misra (2008), Ahmed *et al*. (2009), Zafaryab *et al*. (2010) and Shehzad *et al*. (2011) developed some series to obtain minimal CGNDs for specific cases. Iqbal *et al*. (2012) presented minimal CGNDs only for k = 3. Ahmed and Akhtar (2011b), Ahmed *et al*.(2013) and Ollis (2016) constructed generalised neighbour designs for some cases of linear blocks. In this article, generators are developed through a method of cyclic shifts to obtain minimal CGNDs in blocks of (i) two different sizes, and (ii) three different sizes. In these designs, $3v/2$ unordered pairs of distinct treatments appear twice as neighbours while all others appear once. The rest of the organisation of this article is as follows:

In Section 2, the method of cyclic shifts (Rule I) is explained for the construction of minimal CGNDs. In Section 3, generators are developed for m (mod 4) ≡ 2 to generate the minimal CGNDs in blocks of (i) equal sizes, (ii) two different sizes, and (iii) three different sizes. In Section 4, generators are developed for m (mod 4) ≡ 3 to generate the minimal CGNDs in blocks of (i) equal sizes, (ii) two different sizes, and (iii) three different sizes, where $m = (v-2)/2$.

2. Method of cyclic shifts

The method of cyclic shifts (Rule I) is described here to generate minimal CBNDs and minimal CGNDs. This method was introduced by Iqbal (1991).

Let $S_j = [q_{j1}, q_{j2}, \ldots, q_{j(k_1-1)}]$; be l sets of shifts for blocks of sizes k_1, $S_i = [q_{i1}, q_{i2}, \ldots, q_{i(k_2-1)}]$ be one set for block of size k_2,where $j = 1, 2, \ldots, l$and $1 \le q_{ji} \le v-1$. If each of $1, 2, \ldots, v-1$ appears once in $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(k_1-1)}, (q_{j1}+ q_{j2}+\ldots+q_{j(k_1-1)}) \bmod v, v-q_{j1}, v-q_{j2}, \ldots, v-q_{j(k_1-1)}, v-(q_{j1}+q_{j2}+\ldots+ q_{j(k_1-1)}) \bmod v, q_{i1}, q_{i2}, \ldots, q_{i(k_2-1)}, (q_{i1}+ q_{i2}+\ldots+q_{i(k_2-1)}) \bmod v, v-q_{i1}, v-q_{i2}, \ldots, v-q_{i(k_2-1)}, v-(q_{i1}+q_{i2}+\ldots+q_{i(k_2-1)}) \bmod v]$ then it will provide CBND. If most of $1, 2, \ldots, v-1$ appear (i) once while remaining do not appear, or (ii) once while remaining appear twice then design is minimal CGND.

To construct minimal CGNDs in which $3v/2$ unordered pairs of distinct treatments appear twice as neighbours for $v = 2ik_1+2u_1k_2+2u_2k_3+\ldots+2u_{(n-1)}k_n-2$, the following is the logic behind the method of cyclic shifts (Rule I).

- A = [1, 2, …, m, m+1, m] will provide required minimal CGNDs if the sum of elements in A is divisible by v. If not, replace one or more elements with their complements to make the sum divisible by v, where $m = (v-2)/2$.
- Divide the resultant elements of Ain i groups of size k_1, u_1 sets for k_2, …, and u_{n-1}sets for k_n such that the sum of each group should be divisible of v.
- Delete one element (any) from each group; the resulting will be sets of shifts to generate required minimal CGNDs.

Example 2.1. Following is minimal CGND generated from S_1 = [3,4,7,12,16], S_2 = [8,9] and S_3 = [10,11] for v = 22, k_1 = 6 and k_2 = 3.

Take v (22) blocks for set of shifts S_1. Allocate 0, 1, …, v-1, treatments to the first unit for each block. Add 3 (mod 22) to each element of the first units to get the second unit elements. Similarly add 4 (mod 22) to each element of the second unit to get the third unit elements, and so on, see table 1.

Table 1: Blocks generated from S_1 = [3,4,7,12,16]

Blocks																					
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	0	1	2
7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	0	1	2	3	4	5	6
14	15	16	17	18	19	20	21	0	1	2	3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	0	1	2	3
20	21	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

Take 22 more blocks for S_2 = [8,9] and generate blocks in the similar way as of S_1, see table 2.

Table 2: Blocks generated from S_2 = [8,9]

Blocks																					
23	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
8	9	10	11	12	13	14	15	16	17	18	19	20	21	0	1	2	3	4	5	6	7
17	18	19	20	21	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Take 22 more blocks for S_3 = [10,11] and generate blocks in the similar way as of S_1, see table 3.

Table 3: Blocks generated from S_3 = [10,11]

Blocks																					
45	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**	**61**	**62**	**63**	**64**	**65**	**66**
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
10	11	12	13	14	15	16	17	18	19	20	21	0	1	2	3	4	5	6	7	8	9
21	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Table 1, Table 2 and Table 3 jointly present the minimal CGND for v = 22, k_1 = 6 and k_2 = 3.

3. Minimal CGNDs in which 3*v*/2Unordered Pairs of Distinct Treatments Appear Twice as Neighbours for *m* (mod 4) ≡ 2

In this Section, minimal CGNDs are constructed in blocks of two and three different sizes for *m* (mod 4) ≡ 2.

Theorem 3.1: If *m* (mod 4) ≡ 2 then sets of shifts derived from A = [1, 2, …, *m*/2, (*m*+4)/2,…, *m*, (*m*+1), (*m*+2), (3*m*+2)/2] will provide minimal CGNDs in which 3*v*/2 unordered pairs of distinct treatments appear twice as neighbours while all others appear once for:

- $v = 2ik-2$ using *i* sets of shifts.
- $v = 2ik_1+2u_1k_2-2$ using *i* sets for k_1 and $u_1$sets for k_2.
- $v = 2ik_1+2u_1k_2+2u_2k_3-2$ using *i* sets for k_1, $u_1$sets for k_2 and u_2 sets for k_3, and
- Similarly in blocks of four, five, …, *n* different sizes.

3.1 Minimal CGNDs in blocks of two different sizes for *m* (mod 4) ≡ 2

Generator 3.1.1. Minimal CGNDs can be constructed from the *i* sets of shifts fork1 and two fork2 using theorem 3.1 for $v = 2(i+2)k1-6$, k1 = $4l+2$, k2 = k1-1, *i* odd, *l* integer,*m* (mod 4) ≡ 2 and *m* = (*m*-2)/2.

Example 3.1.1. Minimal CGND is constructed from the following sets for*v* = 30, k1 = 6, k2 = 5.

S1=[2,3,4,6,14], S2=[9,10,12,22], S3 =[11,13,15,16]

Generator 3.1.2. Minimal CGNDs can be constructed from the *i* sets of shifts for$k_1$and two for $k_2$using theorem 3.1 for $v = 2(i+2)k_1-6$, k_1(odd) > 3, $k_2= k_1-1$, *i* (mod 4) ≡ 0, *m* (mod 4) ≡ 2 and *m* = (*m*-2)/2.

Example 3.1.2. Minimal CGND is constructed from the following sets for *v* = 54, $k_1= 5$, $k_2= 4$.

S_1=[3,4,18,27], S_2=[8,9,10,22], S_3=[20,23,24,28]
S_4=[17,19,26,40], S_5=[11,15,21], S_6=[12,16,25]

Generator 3.1.3. Minimal CGNDs can be constructed from the *i* sets of shifts for$k_1$and two for $k_2$using theorem 3.1 for $v = 2(i+2)k_1-10$, $k_1= 4l$, $k_2= k_1-2$, *l* (integer) > 1, *i* integer, *m* (mod 4) ≡ 2 and *m* = (*m*-2)/2.

Example 3.1.3. Minimal CGND is constructed from the following sets of shifts for $v = 38$, $k_1= 8$ and $k_2= 6$.

S_1=[2,3,4,5,6,8,9], S_2=[11,12,13,14,19], S_3=[16,17,20,28,18]

Generator 3.1.4. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 3.1 for $v = 2(i+2)k_1-10$, $k_1= 4l+2$, $k_2= k_1-2$, i even, l integer, $m \pmod 4) \equiv 2$ and $m = (m-2)/2$.

Example 3.1.4. Minimal CGND is constructed from the following sets for $v = 38$, $k_1= 6$, $k_2= 4$.

S_1=[2,3,6,8,18], S_2=[9,13,14,16,19], S_3=[17,20,28], S_4=[7,12,15]

Generator 3.1.5. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 3.1 for $v = 2(i+2)k_1-10$, $k_1(\text{odd}) > 3$, $k_2= k_1-2$, $i \pmod 4) \equiv 2$, $m \pmod 4) \equiv 2$ and $m = (m-2)/2$.

Example 3.1.5. Minimal CGND is constructed from the following sets of shifts for $v = 30$, $k_1= 5$ and $k_2= 3$.

S_1=[3,4,5,16]. S_2=[9,10,12,22], S_3=[11,13], S_4=[14,15]

Generator 3.1.6. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 3.1 for $v = 2(i+2)k_1-14$, $k_1= 4l+2$, $k_2= k_1-3$, i odd, l integer, $m \pmod 4) \equiv 2$ and $m = (m-2)/2$.

Example 3.1.6. Minimal CGND is constructed from the following sets for $v = 22$, $k_1= 6$, $k_2= 3$.

S_1=[3,4,7,12,16], S_2=[8,9], S_3=[10,11]

Generator 3.1.7. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 3.1 for $v = 2(i+2)k_1-14$, $k_1(\text{odd}) > 3$, $k_2= k_1-3$, $i \pmod 4) \equiv 0$, $m \pmod 4) \equiv 2$ and $m = (m-2)/2$.

3.2 Minimal CGNDs in blocks of three different sizes for *m* (mod 4) ≡ 2

Generator 3.2.1. Minimal CGNDs can be constructed from *i* sets of shifts for k_1, two for k_2 and one for k_3 using theorem 3.1 for $v = 2(i+3)k_1-10$, $k_1= 4l$, $l > 1$, *i* integer, $k_2= k_1-1$, $k_3= k_1-2$, m (mod 4) ≡ 2 and$m = (v-2)/2$.

Example 3.2.1. Minimal CGND is constructed from the following sets for $v = 54$, $k_1= 8$, $k_2= 7$ and $k_3= 6$.

S_1=[2,3,4,5,6,7,26], S_2=[9,11,12,13,15,40],
S_3=[20,21,22,25,27,28], S_4=[16,17,18,23,24]

Generator 3.2.2. Minimal CGNDs can be constructed from *i* sets of shifts for k_1, two for k_2 and one for k_3 using theorem 3.1 for $v = 2(i+3)k_1-10$, $k_1= 4l+2$, l integer, $k_2= k_1-1$,$k_3= k_1-2$, *i* odd, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Example 3.2.2. Minimal CGND is constructed from the following sets for $v = 38$, $k_1= 6$, $k_2= 5$ and $k_3= 4$.

S_1=[2,4,5,6,20], S_2=[9,12,19,28], S_3=[14,15,16,18], S_4=[7,11,17]

Generator 3.2.3. Minimal CGNDs can be constructed from *i* sets for k_1, two sets of shifts for k_2 and two for k_3 using theorem 3.1 for $v = 2(i+4)k_1-14$, $k_1= 4l+2$, l integer, $k_2= k_1-1$, $k_3= k_1-2$, *i* odd, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Example 3.2.3. Minimal CGND is constructed from the following sets for $v = 46$, $k_1= 6$, $k_2= 5$ and $k_3= 4$.

S_1=[9,10,22,21,24], S_2=[5,7,14,17], S_3=[15,16,19,34],
S_4=[2,20,23], S_5=[11,13,18]

Generator3.2.4. Minimal CGNDs can be constructed from *i* sets of shifts for k_1, two for k_2 and two for k_3 using theorem 3.1 for v=$2(i+4)k_1$-14,k_1(odd)>3, $k_2= k_1-1$, $k_3= k_1-2$, *i* (mod 4) ≡ 2, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Example 3.2.4. Minimal CGND is constructed from the following sets for $v = 46$, $k_1= 5$, $k_2= 4$ and $k_3= 3$.

S_1 = [3,4,13,24], S_2 = [11,18,19,34], S_3 = [9,14,17],

S_4= [8,15,16], S_5= [20,21], S_6= [22,23]

Generator 3.2.5. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2, two for k_3 using theorem 3.1 for $v = 2(i+3)k_1-12$, $k_1 (\text{mod } 4) = 1$, $k_2= k_1-1$, $k_3= k_1-2$, i (mod 4) ≡ 2, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Example 3.2.5. Minimal CGND is constructed from the following sets for $v = 38$, $k_1= 5$, $k_2= 4$ and $k_3= 3$.

S_1= [3,4,9,20], S_2= [12,14,17,28], S_3= [8,11,13], S_4= [15,16], S_5= [18,19]

Generator 3.2.6. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2, two for k_3 using theorem 3.1 for $v = 2(i+3)k_1-12$,$k_1(\text{mod } 4) = 3$, $k_2= k_1-1$, $k_3= k_1-2$, i (mod 4) ≡ 0, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Generator 3.2.7. Minimal CGNDs can be constructed from i sets of shifts fork_1,two each fork_2, k_3 using theorem 3.1 for $v = 2(i+4)k_1-18$, $k_1= 4l$, $k_2= k_1-1$, $k_3= k_1-3$, l &i integer, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Generator 3.2.8. Minimal CGNDs can be constructed from i sets of shifts fork_1,two each fork_2 ,k_3 using theorem 3.1 for $v = 2(i+4)k_1-18$, $k_1= 4l+2$,k_2=k_1-1,k_3=k_1-3,$i$even,m(mod4)≡ 2,m=(v-2)/2and$l$integer.

Generator3.2.9. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for $k_2$and one for k_3 using theorem 3.1 for$v$=2(i+3)k_1-12,k_1(odd)>3, $k_2= k_1-1$, $k_3= k_1-3$, i (mod 4) ≡ 1, m (mod 4) ≡ 1 and $m = (v-2)/2$.

Example 3.2.9. Minimal CGND is constructed from the following sets for$v = 44$, $k_1= 7$, $k_2= 6$ and $k_3= 4$.

S_1=[4,7,13,16,22,23], S_2=[9,10,11,12,38],
S_3=[14,15,17,18,19], S_4=[2,20,21]

Generator3.2.10. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2 and two for k_3 using theorem 3.1for$v$=2(i+3)k_1-16,k_1(odd)>3, $k_2= k_1-1$, $k_3= k_1-3$, i(mod 4) ≡ 3, m (mod 4) ≡ 1 and $m = (v-2)/2$.

Generator 3.2.11. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2, one for k_3 using theorem 3.1 for $v = 2(i+3)k_1-12$, k_1(mod 4) ≡ 1, $k_2= k_1-1$,$k_3= k_1-3$, i (mod 4) ≡ 2, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Generator 3.2.12. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2 and one for k_3 using theorem 3.1 for $v = 2(i+3)k_1-12$, $k_1(\text{mod } 4) \equiv 3$, $k_2= k_1-1$, $k_3= k_1-3$, i (mod 4) ≡ 0, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Generator 3.2.13. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2, two for k_3 using theorem 3.1 for $v = 2(i+3)k_1-16$, $k_1(\text{mod } 4) \equiv 3$, $k_2= k_1-1$, $k_3= k_1-3$, i (mod 4) ≡ 2, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Generator 3.2.14. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2 and two for k_3 using theorem 3.1 for $v = 2(i+3)k_1-18$, $k_1= 4l$,l &i integer, $k_2= k_1-2$, $k_3= k_1-3$, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Generator 3.2.15. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2, two for k_3 using theorem 3.1 for $v = 2(i+3)k_1-18$, $k_1= 4l+2$, $k_2=k_1-2$,$k_3=k_1-3$,$i$odd,m(mod4)≡ 2,$m=(v-2)/2$and$l$integer.

Example 3.2.15. Minimal CGND is constructed from the following sets for$v = 30$, $k_1= 6$, $k_2= 4$ and $k_3= 3$.

S_1= [3,4,5,6,10], S_2= [13,16,22], S_3= [11,12], S_4= [14,15]

Generator 3.2.16. Minimal CGNDs can be constructed from i sets of shifts fork_1,two each fork_2, k_3 using theorem 3.1 for $v = 2(i+4)k_1-22$, $k_1= 4l+2$, $k_2=k_1-2$,$k_3=k_1-3$,m(mod4)≡ 2,$i$odd,$l$integer and $m=(v-2)/2$.

Generator3.2.17. Minimal CGNDs can be constructed from i sets of shifts fork_1,two each for$k_2$and k_3 using theorem 3.1 for $v=2(i+4)k_1-22$,k_1(odd)>3, $k_2= k_1-2$, $k_3= k_1-3$, i(mod 4) = 2, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Generator 3.2.18. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2, one for k_3 using theorem 3.1 for $v = 2(i+3)k_1-16$, $k_1(\text{mod } 4) \equiv 3$, $k_2= k_1-2$, $k_3= k_1-3$, i (mod 4) ≡ 2, m (mod 4) ≡ 2, $m = (v-2)/2$.

4. Minimal CGNDs in which 3*v*/2 Unordered Pairs of Distinct Treatments Appear Twice as Neighbours for *m* (mod 4) ≡ 1

In this Section, minimal CGNDs are constructed in blocks of two and three different sizes for m (mod 4) ≡ 1.

Theorem 4.1: If m (mod 4) ≡ 1 then sets of shifts derived from A = [1, 2,…, (m-1)/4, (m+7)/4,…, m, (m+1), (m+2), (7m+5)/4] will provide minimal CGNDs in which 3v/2 unordered pairs of distinct treatments appear twice as neighbours while all others appear once for:

- $v = 2ik-2$ using i sets of shifts.
- $v = 2ik_1+2u_1k_2-2$ using i sets for k_1 and u_1 sets for k_2.
- $v = 2ik_1+2u_1k_2+2u_2k_3-2$ using i sets for k_1, u_1 sets for k_2 and u_2 sets for k_3, and
- Similarly in blocks of four, five, …, n different sizes.

4.1 Minimal CGNDs in blocks of two different sizes for m (mod 4) ≡ 1

Generator 4.1.1. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 4.1 for $v = 2(i+2)k_1-6$, k_1 (mod 4) ≡ 1, $k_2 = k_1-1$, i (mod 4) ≡ 3, m (mod 4) ≡ 1 and $m = (m-2)/2$.

Example 4.1.1. Minimal CGND is constructed from the following sets for $v = 44$, $k_1 = 5$, $k_2 = 4$.

S_1=[3,4,14,22], S_2=[9,10,23,38], S_3 =[17,18,19,21],
S_4=[11,12,16], S_5 =[7,15,20]

Generator 4.1.2. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 4.1 for $v = 2(i+2)k_1-6$, k_1(mod 4) ≡ 3, $k_2 = k_1-1$, i (mod 4) ≡ 1, m (mod 4) ≡ 1 and $m = (m-2)/2$.

Example 4.1.2. Minimal CGND is constructed from the following sets for $v = 36$, $k_1 = 7$, $k_2 = 6$.

S_1=[3,4,6,7,31,19], S_2=[10,11,12,18,13], S_3=[9,14,15,16,17]

Generator 4.1.3. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 4.1 for $v = 2(i+2)k_1-10$, k_1(mod 4) ≡ 1, $k_2 = k_1-2$, i (mod 4) ≡ 1, m (mod 4) ≡ 1 and $m = (m-2)/2$.

Example 4.1.3. Minimal CGND is constructed from the following sets for $v = 20$, $k_1 = 5$, $k_2 = 3$.

S_1= [4,6,11,17], S_2=[7,8], S_3=[9,10]

Generator 4.1.4. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 4.1 for $v = 2(i+2)k_1-10$, k_1(mod 4) ≡ 3, $k_2 = k_1-2$, i (mod 4) ≡ 3, m (mod 4) ≡ 1 and $m = (m-2)/2$.

Generator 4.1.5. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 4.1 $v = 2(i+2)k_1-14$, $k_1 (\text{mod } 4) \equiv 1$, $k_2 = k_1-3$, i (mod 4) ≡ 3, m (mod 4) ≡ 1 and $m = (m-2)/2$.

Generator 4.1.6. Minimal CGNDs can be constructed from the i sets of shifts for k_1 and two for k_2 using theorem 4.1 for $v = 2(i+2)k_1-14$, $k_1 (\text{mod } 4) \equiv 3$, $k_2 = k_1-3$, i (mod 4) ≡ 1, m (mod 4) ≡ 1 and $m = (m-2)/2$.

Example 4.1.6. Minimal CGND is constructed from the following sets for $v = 28$, $k_1 = 7$, $k_2 = 4$.
S_1=[7,8,10,14,15,24], S_2=[5,9,11], S_3=[2,12,13]

4.2 Minimal CGNDs in blocks of three different sizes for m (mod 4) ≡ 1

Generator 4.2.1. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2 and two for k_3 using theorem 4.1 for $v = 2(i+3)k_1-12$, $k_1 = 4l+2$, l integer, $k_2 = k_1-1$, $k_3 = k_1-2$, i odd, m (mod 4) ≡ 1 and $m = (v-2)/2$.

Example 4.2.1. Minimal CGND is constructed from the following sets for $v = 36$, $k_1 = 6$, $k_2 = 5$ and $k_3 = 4$.

S_1=[4,6,11,31,19], S_2=[14,15,16,17], S_3=[9,12,13], S_4=[7,8,18]

Generator 4.2.2. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2 and two for k_3 using theorem 4.1 for $v = 2(i+3)k_1-12$, $k_1 = 4l$, $l > 1$, $k_2 = k_1-1$, $k_3 = k_1-2$, i integer, m (mod 4) ≡ 1 and $m = (v-2)/2$.

Generator 4.2.3. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2 and one for k_3 using theorem 4.1 for $v = 2(i+3)k_1-10$, k_1(odd) > 3, $k_2 = k_1-1$, $k_3 = k_1-2$, i (mod 4) ≡ 1, m (mod 4) ≡ 2 and $m = (v-2)/2$.

Example 4.2.3. Minimal CGND is constructed from the following sets for $v = 30$, $k_1 = 5$, $k_2 = 4$ and $k_3 = 3$.

S_1=[6,11,16,22], S_2=[3,12,13], S_3=[7,9,10], S_4=[14,15]

Generator4.2.4. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2 and two for k_3 using theorem 4.1 for $v=2(i+3)k_1-12$, k_1(odd)>3, $k_2 = k_1-1$, $k_3 = k_1-2$, $m = (v-2)/2$, i (mod 4) ≡ 1 and m (mod 4) ≡ 1.

Example 4.2.4. Minimal CGND is constructed from the following sets for$v$ = 28, k_1= 5, k_2= 4 and k_3= 3.

S_1=[5,9,15,24], S_2=[6,8,12], S_3=[10,11], S_4=[13,14]

Generator4.2.5. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2 and one for k_3 using theorem 4.1 for$v$=2(i+3)k_1-10,k_1(mod4)=1, k_2= k_1-1, k_3= k_1-2, i (mod 4) ≡ 0,m (mod 4) ≡ 1 and m = (v-2)/2.

Generator 4.2.6. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2, two for k_3 using theorem 4.1 for v= 2(i+4)k_1-14,k_1(mod 4) =1, k_2= k_1-1, k_3= k_1-2, i (mod 4) ≡ 1,m (mod 4) ≡ 1 and m = (v-2)/2.

Generator 4.2.7. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2, one for k_3 using theorem 4.1 for v= 2(i+3)k_1-10, k_1(mod 4) = 3, k_2= k_1-1, k_3= k_1-2, i (mod 4) ≡ 2,m (mod 4) ≡ 1 and m = (v-2)/2.

Generator 4.2.8. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for $k_2$and two for k_3 for v = 2(i+4)k_1-14,k_1(mod 4) = 3, k_2= k_1-1, k_3= k_1-2, i (mod 4) ≡ 3,m (mod 4) ≡ 1 and m = (v-2)/2.

Generator 4.2.9. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2 and one for k_3 using theorem 4.1 for v = 2(i+3)k_1-12, k_1= 4l, k_2= k_1-1, k_3= k_1-3,m (mod 4) ≡ 1, m = (v-2)/2,i and l integer.

Generator 4.2.10. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2 and one for k_3 using theorem 4.1 for v = 2(i+3)k_1-12, k_1= 4l+2, k_2= k_1-1, k_3= k_1-3, m (mod 4) ≡ 1, m = (v-2)/2,i odd and l integer.

Generator 4.2.11. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2, two for k_3 using theorem 4.1 for v = 2(i+3)k_1-16, k_1= 4l+2, k_2=k_1-1,k_3=k_1-3,$i$even,m(mod4)≡1,m=(v-2)/2and$l$integer.

Generator4.2.12. Minimal CGNDs can be constructed from i sets of shifts fork_1,two each for$k_2$and k_3 using theorem 4.1 for v=2(i+4)k_1-18,k_1(odd)>3, k_2= k_1-1, k_3= k_1-3, i(mod 4) = 0,m (mod 4) ≡ 2 and m = (v-2)/2.

Generator 4.2.13. Minimal CGNDs can be constructed from i sets of shifts for k_1, two each for k_2 and k_3 using theorem 4.1 for v = 2(i+4)k_1-18, k_1(mod 4) ≡ 3, k_2= k_1-1, k_3= k_1-3, i(mod 4) ≡ 1,m (mod 4) ≡ 1 and m = (v-2)/2.

Generator 4.2.14. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2 and one for k_3 using theorem 4.1 for $v = 2(i+3)k_1-16$, $k_1= 4l+2$, $k_2= k_1-2$, $k_3= k_1-3$, $m \pmod 4 \equiv 1$, $m = (v-2)/2$, i even, and l integer.

Generator4.2.15. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2 and one for $k_3$using theorem 4.1 for $v=2(i+3)k_1-16$,k_1(odd)>3, $k_2= k_1-2$, $k_3= k_1-3$, $i \pmod 4 \equiv 3$,$m \pmod 4 \equiv 1$ and $m = (v-2)/2$.

Generator4.2.16. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2 and two for k_3 using theorem 4.1 for$v=2(i+3)k_1-16$,k_1(odd)>3, $k_2= k_1-2$, $k_3= k_1-3$, $i \pmod 4 \equiv 3$,$m \pmod 4 \equiv 1$ and $m = (v-2)/2$.

Generator4.2.17. Minimal CGNDs can be constructed from i sets of shifts for k_1, two for k_2, two for k_3 using theorem 4.1 for $v= 2(i+4) k_1-22$,$k_1 \pmod 4 =1$, $k_2= k_1-2$,$k_3= k_1-3$, $i \pmod 4 \equiv 1$, $m \pmod 4 \equiv 1$ and $m = (v-2)/2$.

Generator 4.2.18. Minimal CGNDs can be constructed from i sets of shifts for k_1, one for k_2 and two for k_3 using theorem 4.1 for $v = 2(i+3)k_1-18$, $k_1 \pmod 4 \equiv 3$, $k_2= k_1-2$, $k_3= k_1-3$, $i \pmod 4 \equiv 2$,$m \pmod 4 \equiv 1$, $m = (v-2)/2$.

Generator 4.2.19. Minimal CGNDs can be constructed from i sets of shifts for k_1, two each for $k_2$and k_3 using theorem 4.1 for $v = 2(i+4)k_1-22$, $k_1 \pmod 4 \equiv 3$, $k_2= k_1-2$,$k_3= k_1-3$, $i \pmod 4 \equiv 3$,$m \pmod 4 \equiv 1$ and $m = (v-2)/2$.

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