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Analytical solutions of modified Friedmann equation in Tsallis Cosmology for nonflat universe

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Abstract

The modified Friedmann equation discussed in this paper is the equation derived by Sheykhi (Sheykhi, 2018) in Tsallis cosmology. Tsallis cosmology is a cosmological model developed by Tsallis and Cirto (Tsallis & Cirto, 2013) based on the thermodynamic entropy of a gravitational system such as a black hole. Syiekhi has provided a solution to the equation for the case of flat universe $k = 0$, but in this paper, we worked on the solutions to the equation for the case of nonflat universe with $k \neq 0$. We obtained solutions in the form of hypergeometric functions for the era of matter domination and the era of radiation domination, which are only distinguished by constants. For this reason, at the end, we declare the equation solution for both eras in one form of a general solution. In this paper we also provide examples of specific solutions for matter domination era, as an example case, that is derived from the general solution that has been obtained.

Keywords: *friedmann equation; Tsallis Cosmology; analytical solution; nonflat universe*

Introduction

The Friedmann equation is an equation that plays a role in explaining a non-static universe or a large-scale homogeneous and isotropic universe. Einstein constructed a static universe model which the universe could not expand. Therefore, in the field equations obtained, the cosmological constant added into the equation. However, this was different from Friedmann's proposal, several years later, Einstein put forward the supposed natural model. In 1922, Friedmann proposed that the universe was expanding, and finally the hypothesis emerged that, in early time, the universe came from the same point (Alpher & Herman, 1949; Gamow, 1946; Price, 2012; Turok, 1983). This model is in line with the results of observations carried out by Hubble and Humason in 1929 which showed that all galaxies move away from observers at distant speeds in proportion to a proportional constant called the Hubble constant (Hubble, 1929). Furthermore, in 1965, Penzias and Wilson observed some anisotropy of cosmic wave background radiation (Penzias & Wilson, 1965), which not only supported the Friedmann model, but also opened up new ways of looking at the early history of the universe.

Mathematically, the Friedmann equation derived from the Einstein Field Equation, especially from the FLRW metric (Friedmann-Lamaitre-Robertson-Walker) (Amendola & Tsujikawa, 2010). This metric contains a time-dependent function that gives a picture of the expanding universe model. This model also gives the basis of the standard big bang universe model which states that the universe originates from a very high and compressed energy density state in very small scale. The Friedmann equation can be explicitly seen in various sources, such as (Amendola & Tsujikawa, 2010; Bonometto, Gorini, & Moschella, 2001; Cheng, 2009; Ryder, 2009). The Friedmann model has been able to describe the current model of the hyperbolic open universe, with a very small curvature. Furthermore, this model is also able to estimate the age of the universe based on physical variables such as the initial density of the universe (not at the singularity point), temperature, expansion speed of the universe, and several other physical variables (Hidayat, 2010).

Today, cosmology is supported by fields such as high-energy astrophysics, particle physics, quantum gravitational mechanics, improved observational astronomy using robotic telescopes, and various other fields that also support the development of increasingly modern models of the universe and more 'real'. Therefore, the Friedmann equation has experienced rapid development in various forms, such as (Ambjørn & Watabiki, 2017; Bertolami & Páramos, 2014; Coquereaux, 2015; Hikmawan, Soda, Suroso, & Zen, 2016; Pardede, Suroso, & Zen, 2018; Sheykhi, 2018). One of the interesting cosmological models today is the Tsallis cosmology model (Sheykhi, 2018; Tsallis & Cirto, 2013). This model is based on the state of entropy of a gravitational source. Tsallis (Tsallis & Cirto, 2013) generalized standard thermodynamics to a more general form called the Non-Extensive model and can be applied to all cases, and which still used the Boltzmann-Gibbs Theory as a limitation. Therefore, Bolzmann-Gibbs's additive entropy must be generalized to a non-extensive form (Lyra & Tsallis, 1998), namely non-additive entropy (called the Tsallis entropy) which states that the entropy of a system does not have to be the sum of the entropy of each sub-system. There have been several studies that have examined the cosmology of Tsallis in the past few years (Barboza Jr, Nunes, Abreu, & Neto, 2015; Ghaffari, Moradpour, Bezerra, Graça, & Lobo, 2019; Lymperis & Saridakis, 2018; Soares, Barboza Jr, Abreu, & Neto, 2019).

The Friedmann equation was originally derived from field equations, but nowadays, there have been those which derive it from the thermodynamic law regarding entropy. Most recently, Sheykhi (Sheykhi, 2018) has examined the Friedmann equation which was derived based on Tsallis cosmology whose main study was from thermodynamic reviews. In his work, the Friedmann equation is derived based on Tsallis non-extensive entropy. Furthermore, they analyzed the implications of the Friedmann equation into the era of matter domination and the era of radiation domination. However, in his work, he only analyzed and found solutions to the Friedmann equation for a flat universe, $k = 0$, but not for $k \neq 0$. Therefore, in this paper, we are working on the solutions to the equation for a nonflat universe state to review the same equation with Sheykhi. The solutions that we have obtained in the end are also applied to two eras, namely the era of matter domination and the era of radiation domination.

The structure of this paper is: first, introduction, which contains the basic reasons for this research; second, about the analytical solutions of the modified Friedmann equation, and finally, conclusion and discussion. Natural scale has been chosen where $k_B = c = \hbar = 1$; where k_B is Boltzmann constant; c is the speed of light in a vacuum; and $\hbar$ is the Plank constant divided by 2π.

Analytical Solutions of Modified Friedmann Equation

The modified Friedmann equation studied in this paper coming from Tsalis Cosmological model (Sheykhi, 2018),

$$\left(H^2 + \frac{k}{a^2}\right)^{2-\beta} = \frac{8\pi L_p^2}{3}\rho \tag{1}$$

and

$$(4-2\beta)\,\frac{\ddot{a}}{a} + (2\beta-1)\left(H^2 + \frac{k^2}{a^2}\right)^{2-\beta} = -8\pi L_p^2\,, \tag{2}$$

where $a = a(t)$ is the scale factor of universe; $\ddot{a}$ is the second time derivative of scale factor; $H \equiv \dot{a}/a$ is Hubble Parameter; $\dot{a}$ is the first derivative of scale factor; $L_p^2 = 1/(4\gamma)$; γ is a constant; β is Tsallis parameter; p is pressure; ρ is density; k is a constant that appears from the following FLRW metric

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -dt^2 + a^2 d\sigma^2\ , \tag{3}$$

where $\mu, \nu = 0,1,2,3$; $g_{\mu\nu}$ is a metric tensor; $d\sigma^2$ is a time-dependent three-dimensional space metric with a curvature constant k. Metric $d\sigma^2$ can be expressed in equation

$$d\sigma^2 = h_{ij}\, dx^i\, dx^j = \frac{dr^2}{1-kr^2} \quad + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{4}$$

where $(x^1, x^2, x^3) = (r, \theta, \phi)$; with $h_{11} = (1-kr^2)^{-1}$; $h_{22} = r^2$; and $h_{33} = r^2 \sin\theta$. Equation (3) is a metric for derive equations (1) and (2). These two equations are nonlinear ordinary differential equations.

If equation (1) and (2) are combined, the second derivative equation of the scale factor is obtained as stated in (Sheykhi, 2018) is

$$\frac{\ddot{a}}{a} = -\frac{4\pi\, L_p^2 p}{3(2-\beta)} \; ((2\beta - 1)\rho + 3p). \tag{5}$$

Because our universe expands, it is clear that $\ddot{a} > 0$, and it has also been shown that the equation (2) will return to the standard cosmological form if $\beta = 1$ and $\omega = -1/3$; where ω is a state parameter with $\omega = p/\rho$. Whereas, the equation which states the relationship between ρ, p, and H is called a continuity equation,

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{6}$$

Next, we look for a solution to the equation (1). To find a solution to this equation, we divide it into two cases, namely the era of matter domination and the era of radiation domination.

Matter domination era

Era of matter domination is stated by $p \ll \rho$, so we can take $p = 0$ for simplification. Therefore, the form of equation (6) becomes

$$\dot{\rho} + 3H\rho = 0, \tag{7}$$

or in a more explicit form

$$\frac{d\rho}{dt} = -3\frac{\dot{a}}{a}\, \rho, \tag{8}$$

where $\dot{a} = da/dt$, then the solution to equation (8) is

$$\rho(t) = \rho_0 \frac{a_0^3}{a(t)^3}, \tag{9}$$

where ρ_0 and a_0 respectively are the initial density and scale factors ($t = 0$), while $\rho(t)$ and $a(t)$ are respectively the density and scale factors at the time of t. Equations (9) is entered into equation (1) and keeping in mind that $H = \dot{a}/a$, is obtained

$$\left(\frac{\dot{a}^2}{a} + \frac{k}{a^2}\right)^{2-\beta} = 8\pi\, L_p^2\, \rho_0 \frac{a_0^3}{3a^3}. \tag{10}$$

By doing some algebraic steps, the following equation is obtained

$$\frac{da}{dt} = \left(C_1^{\frac{1}{2-\beta}} a^{\frac{1-2\beta}{2-\beta}} - k\right)^{1/2}, \tag{11}$$

where

$$C_1 = 8\pi\, L_p^2 \rho_0 \frac{a_0^3}{3}. \tag{12}$$

Equation (11) can be simplified into equation

$$\frac{da}{dt} = \sqrt{Aa^m - k}, \tag{13}$$

where $A = C_1^{\frac{1}{2-\beta}}$, and $m = \frac{1-2\beta}{2-\beta}$. Note that, this equation can be solved in two cases, namely at $k = 0$ or flat universe and $k \neq 0$ for nonflat universe. Especially for $k = 0$, the equation (13) changes to the form $da/dt = A^{\frac{1}{2}} a^{m/2}$ which the solution can be easily obtained, and has been done in (Sheykhi, 2018).

Next, we look for a solution of the equation (13) for $k \neq 0$. The equation (13) can also be expressed in the equation

$$\frac{\frac{da}{dt}}{\sqrt{Aa^m - k}} = 1. \tag{14}$$

Both sides of the equation are integrated to t,

$$\int \frac{\frac{da}{dt}}{\sqrt{Aa^m - k}} \quad dt = \int dt. \tag{15}$$

The integral result of equation (15) is

$$\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{m};\frac{1}{m}+1;\frac{Aa^m}{k}\right)}{\sqrt{Aa^m - k}}\sqrt{1-\frac{Aa^m}{k}}a = t + C. \tag{16}$$

Equation (16) is a general solution of the matter domination era for the case of $k \neq 0$, where ${}_2F_1$ is *hypergeometric function*, has the following explicit statement

$${}_2F_1(\alpha_1, \alpha_2; \alpha_3\,; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\,(\alpha_2)_n}{(\alpha_3)_n}\frac{z^n}{n!} \tag{17}$$

is undefined for α_3 non-positive integers. The symbol $(\alpha_i)_n$ is *Pochhammer symbol* which has a definition

$$(\alpha_i)_n = \begin{cases} 1 & ;\ n=0 \\ \alpha_i\ (\alpha_i+1)\ldots(\alpha_i+n-1) & ;\ n>0 \end{cases} \tag{18}$$

For $n > 0$, *Pochhammer* symbols in equation (17) can be expressed in form

$$\left(\frac{1}{2}\right)_n = \prod_{j=1}^{n}\left(j-\frac{1}{2}\right), \tag{19}$$

$$\left(\frac{1}{m}\right)_n = \prod_{j=1}^{n}\left(\frac{1}{m}+(j-1)\right), \tag{20}$$

$$\left(1+\frac{1}{m}\right)_n = \prod_{j=1}^{n}\left(j+\frac{1}{m}\right), \tag{21}$$

where j is positive integer. From the forms of the *Pochhammer* symbols then for hypergeometric functions in the equation (16) can be expressed in form

$${}_2F_1\left(\frac{1}{2},\frac{1}{m};\frac{1}{m}+1;\frac{Aa^m}{k}\right) = 1 + \left(\frac{A}{k}\right)^n \sum_{n=1}^{\infty}\left(\prod_{j=1}^{n}\left(j-\frac{1}{2}\right)\right)\frac{a^{mn}}{(1+mn)(n!)}\ . \tag{22}$$

To be able to see the behavior and explicit expression of the solution (17), the approximation to the second order of this hypergeometric function is carried out in this paper. For an approximation of first order ($n = 1$) obtained

$${}_2F_1|_n = 1 + \frac{A}{2k}\frac{a^m}{1+m}\ , \tag{23}$$

while the second order approximation ($n = 2$) is obtained

$$ {}_2F_1|_{n=2} = 1 + \frac{3}{4}\left(\frac{A}{k}\right)^2 \left(\frac{a^m}{1+m} + \frac{a^{2m}}{(1+2m)(2!)}\right). \tag{24}$$

So, we give some examples of specific solutions to equations (16) based on (23) and (24). For an approximation of first order at $m = 0$, the solution is

$$a(t) = \frac{2k\sqrt{A-k}}{\sqrt{1-\frac{A}{k}(A+2k)}}(t + C), \tag{25}$$

Also, for $m = 1$, the solutions are

$$a_{1,2}(t) = \frac{-2k \mp 2\sqrt{k^2 - iA^{\frac{3}{2}}\ (t+C)}}{A} \tag{26}$$

and

$$a_{3,4}(t) = \frac{-2k \mp 2\sqrt{k^2 + iA^{\frac{3}{2}}(t+C)}}{A}, \tag{27}$$

with $A \neq 0$.

Next, for an approximate second order, at $m = 0$, that is

$$a(t) = \frac{8k^2\sqrt{A-k}}{\sqrt{1-\frac{A}{k}(3A^2+8k^2)}}\ (t + C). \tag{28}$$

For an approximate of second order at $m = 1$, the solutions are

$$a_1(t) = -1 + \frac{B}{3^{\frac{1}{3}}\left(f_1(t)+2\sqrt{6}\sqrt{g_1(t)}\right)^{\frac{1}{3}}} + \frac{\left(f_1(t)+2\sqrt{6}\sqrt{g_1(t)}\right)^{\frac{1}{3}}}{3^{2/3}A^2}, \tag{29}$$

$$a_2(t) = \frac{1}{6}\left(-6 - \frac{3^{\frac{2}{3}}(\sqrt{3}i+1)B}{\left(f_1(t)+2\sqrt{6}\sqrt{g_1(t)}\right)^{\frac{1}{3}}} + \frac{3^{\frac{1}{3}}(\sqrt{3}i-1)\left(f_1(t)+2\sqrt{6}\sqrt{g_1(t)}\right)^{\frac{1}{3}}}{A^2}\right), \tag{30}$$

$$a_3(t) = \frac{1}{6}\left(-6 + \frac{3^{\frac{2}{3}}(\sqrt{3}i-1)B}{\left(f_1(t)+2\sqrt{6}\sqrt{g_1(t)}\right)^{\frac{1}{3}}} - \frac{3^{\frac{1}{3}}(\sqrt{3}i+1)\left(f_1(t)+2\sqrt{6}\sqrt{g_1(t)}\right)^{\frac{1}{3}}}{A^2}\right), \tag{31}$$

$$a_4(t) = -1 + \frac{B}{3^{\frac{1}{3}}\left(f_2(t)+2\sqrt{6}\sqrt{g_2(t)}\right)^{\frac{1}{3}}} + \frac{\left(f_2(t)+2\sqrt{6}\sqrt{g_2(t)}\right)^{\frac{1}{3}}}{3^{\frac{2}{3}}A^2}, \tag{32}$$

$$a_5(t) = \frac{1}{6}\left(-6 - \frac{3^{\frac{2}{3}}(\sqrt{3}i+1)B}{\left(f_2(t)+2\sqrt{6}\sqrt{g_2(t)}\right)^{\frac{1}{3}}} + \frac{3^{\frac{1}{3}}(\sqrt{3}i-1)\left(f_2(t)+2\sqrt{6}\sqrt{g_2(t)}\right)^{\frac{1}{3}}}{A^2}\right), \tag{33}$$

$$a_6(t) = -1 + \frac{(\sqrt{3}i-1)B}{3^{\frac{1}{3}}2\left(f_2(t)+2\sqrt{6}\sqrt{g_2(t)}\right)^{\frac{1}{3}}} - \frac{(\sqrt{3}i+1)\left(f_3(t)+2\sqrt{g_3(t)}\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}6A^2}, \tag{34}$$

where

$$B = 3A^2 - 8k^2, \tag{35}$$

$$f_1(t) = -9A^6 + 36A^4(k^2 - i\,k^{5/2}(C+t)), \tag{36}$$

$$f_2(t) = -9A^6 + 36A^4(k^2 + i\,k^{5/2}(C+t)), \tag{37}$$

$$f_3(t) = -54A^6 + 216A^4k^2 + 216iA^4Ck^{\frac{5}{2}} + 216iA^4k^{5/2}t, \tag{38}$$

and

$$g_1(t) = A^6k^{\frac{5}{2}}(64k^{\frac{7}{2}} + 27i\,A^4(C+t) - 18A^2k^{\frac{3}{2}}(1 + 6i\,C\sqrt{k} + 3C^2k + 6i\sqrt{k}t + 6Ckt + 3kt^2), \tag{39}$$

$$g_2(t) = A^6k^{\frac{5}{2}}(64k^{\frac{7}{2}} - 27i\,A^4(C+t) - 18A^2k^{\frac{3}{2}}(1 - 6i\,C\sqrt{k} + 3C^2k - i\sqrt{k}t + 6Ckt + 3kt^2), \tag{40}$$

$$g_3(t) = (-9A^4 + 24A^2k^2)^3 + 729A^8\left(A^2 - 4ik^2(-i + C\sqrt{k} + \sqrt{k}t)\right)^2. \tag{41}$$

It can be seen from the approximation of first order and second order for $m = 0$ that $\ddot{a}(t) = 0$. This clearly shows that for $m = 0$ or $\beta = 1/2$ it does not match the reality that shows the universe is expanding. This is also in line with Sheykhi's work which gets the value $\beta < 1/2$ (more clearly seen in the results below). While for the first-order approximation for $m = 1$ or $\beta = -1$, $\ddot{a}(t)$ is obtained as follows

$$\ddot{a}_{1,2}(t) = \mp i \frac{\sqrt{A}}{2\left(k^2 - iA^{\frac{3}{2}}(t+C)\right)^{3/2}} \tag{42}$$

and

$$\ddot{a}_{3,4}(t) = \mp i \frac{\sqrt{A}}{2\left(k^2 + iA^{\frac{3}{2}}(t+C)\right)^{3/2}}. \tag{43}$$

Next, for a second-order approximation, we show the six forms of $\ddot{a}(t)$ as follows

$$\ddot{a}_1(t) = \frac{2}{9}\left(\dot{f}_1 + \frac{\sqrt{6}\dot{g}_1}{\sqrt{g_1}}\right)^2\left(\frac{2B}{3^{\frac{1}{3}}F_1^{\frac{7}{3}}} - \frac{1}{3^{\frac{2}{3}}A^2F_1^{\frac{5}{3}}}\right) + \frac{\sqrt{6}}{3}\left(\frac{\ddot{g}_1}{\sqrt{g_1}} - \frac{\dot{g}_1{}^2}{2g_1^{\frac{3}{2}}}\right)\left(\frac{1}{3^{\frac{2}{3}}A^2F_1^{\frac{2}{3}}} - \frac{B}{3^{\frac{1}{3}}F_1^{\frac{1}{3}}}\right), \tag{44}$$

$$\ddot{a}_2(t) = -\frac{2}{9}\left(\dot{f}_1 + \frac{\sqrt{6}\dot{g}_1}{\sqrt{g_1}}\right)^2\left(\frac{(\sqrt{3}i+1)B}{3^{\frac{1}{3}}F_1^{\frac{7}{3}}} + \frac{3^{\frac{1}{3}}(\sqrt{3}i-1)}{6A^2F_1^{\frac{5}{3}}}\right) + \frac{\sqrt{6}}{18}\left(\frac{\ddot{g}_1}{\sqrt{g_1}} - \frac{\dot{g}_1{}^2}{2g_1^{\frac{3}{2}}}\right)\left(\frac{3^{\frac{2}{3}}(\sqrt{3}i+1)B}{F_1^{\frac{4}{3}}} + \frac{3^{\frac{1}{3}}(\sqrt{3}i-1)}{A^2F_1^{\frac{2}{3}}}\right), \tag{45}$$

$$\ddot{a}_3(t) = \frac{2}{9}\left(\dot{f}_1 + \frac{\sqrt{6}\dot{g}_1}{\sqrt{g_1}}\right)^2\left(\frac{(\sqrt{3}i-1)B}{3^{\frac{1}{3}}F_1^{\frac{7}{3}}} + \frac{3^{\frac{1}{3}}(\sqrt{3}i+1)}{6A^2F_1^{\frac{5}{3}}}\right) - \frac{\sqrt{6}}{18}\left(\frac{\ddot{g}_1}{\sqrt{g_1}} - \frac{\dot{g}_1{}^2}{2g_1^{\frac{3}{2}}}\right)\left(\frac{3^{\frac{2}{3}}(\sqrt{3}i-1)B}{F_1^{\frac{4}{3}}} + \frac{3^{\frac{1}{3}}(\sqrt{3}i+1)}{A^2F_1^{\frac{2}{3}}}\right), \tag{46}$$

$$\ddot{a}_4(t) = \frac{2}{9}\left(\dot{f}_2 + \frac{\sqrt{6}\dot{g}_2}{\sqrt{g_2}}\right)^2\left(\frac{2B}{3^{\frac{1}{3}}F_2^{\frac{7}{3}}} - \frac{1}{3^{\frac{2}{3}}A^2F_2^{\frac{5}{3}}}\right) + \frac{\sqrt{6}}{3}\left(\frac{\ddot{g}_2}{\sqrt{g_2}} - \frac{\dot{g}_2^2}{2g_2^{\frac{3}{2}}}\right)\left(\frac{1}{3^{\frac{2}{3}}A^2F_2^{\frac{2}{3}}} - \frac{B}{3^{\frac{1}{3}}F_2^{\frac{1}{3}}}\right), \tag{47}$$

$$\ddot{a}_5(t) = -\frac{2}{9}\left(\dot{f}_2 + \frac{\sqrt{6}\dot{g}_2}{\sqrt{g_2}}\right)^2\left(\frac{(\sqrt{3}i+1)B}{3^{\frac{1}{3}}F_2^{\frac{7}{3}}} + \frac{3^{\frac{1}{3}}(\sqrt{3}i-1)}{6A^2F_2^{\frac{5}{3}}}\right) + \frac{\sqrt{6}}{18}\left(\frac{\ddot{g}_2}{\sqrt{g_2}} - \frac{\dot{g}_2^2}{2g_2^{\frac{3}{2}}}\right)\left(\frac{3^{\frac{2}{3}}(\sqrt{3}i+1)B}{F_2^{\frac{4}{3}}} + \frac{3^{\frac{1}{3}}(\sqrt{3}i-1)}{A^2F_2^{\frac{2}{3}}}\right), \tag{48}$$

$$\ddot{a}_6(t) = \frac{4M}{9}\left(\dot{f}_2 + \frac{\sqrt{6}\dot{g}_2}{\sqrt{g_2}}\right)^2 F_2^{-7/3} - \frac{\sqrt{6}M}{3}\left(\frac{\ddot{g}_2}{\sqrt{g_2}} - \frac{\dot{g}_2^2}{2g_2^{\frac{3}{2}}}\right)F_2^{-\frac{4}{3}} + \frac{2N}{9}\left(\dot{f}_3 + \frac{\dot{g}_3}{\sqrt{g_3}}\right)^2 F_3^{-\frac{5}{3}} - \frac{N}{3}\left(\frac{\ddot{g}_3}{\sqrt{g_3}} - \frac{\dot{g}_3^2}{2g_3^{\frac{3}{2}}}\right)F_3^{-2/3}. \tag{49}$$

where

$$F_1 = F_1(t) = f_1 + 2\sqrt{6g_1}, \tag{50}$$

$$F_2 = F_2(t) = f_2 + 2\sqrt{6g_2}, \tag{51}$$

$$F_3 = F_3(t) = f_3 + 2\sqrt{g_3}, \tag{52}$$

$$\dot{f}_1 = -36iA^4k^{5/2}, \tag{53}$$

$$\dot{f}_2 = 36iA^4k^{5/2}, \tag{54}$$

$$\dot{f}_3 = 216iA^4k^{5/2}, \tag{55}$$

$$\ddot{f}_1 = 0, \tag{56}$$

$$\ddot{f}_2 = 0, \tag{57}$$

$$\ddot{f}_3 = 0, \tag{58}$$

$$\dot{g}_1 = 27A^2k^2(i\sqrt{k}A^8 - 4i - 4\sqrt{k}C - 4\sqrt{k}t), \tag{59}$$

$$\dot{g}_2 = 27A^2k^2(-i\sqrt{k}A^8 + 4i - 4\sqrt{k}C - 4\sqrt{k}t), \tag{60}$$

$$\dot{g}_3 = 5832A^8k^{\frac{5}{2}}(-A^2i + 4k^2i - 4Ck^{\frac{5}{2}} - 4k^{\frac{5}{2}}t), \tag{61}$$

$$\ddot{g}_1 = -108A^2k^{\frac{5}{2}}, \tag{62}$$

$$\ddot{g}_2 = -108A^2k^{\frac{5}{2}}, \tag{63}$$

$$\ddot{g}_3 = -23328A^8k^5, \tag{64}$$

$$M = \frac{(\sqrt{3}i-1)B}{2.3^{\frac{1}{3}}}, \tag{65}$$

$$N = \frac{\sqrt{3}i+1}{6.2^{\frac{1}{3}}A^2}. \tag{66}$$

Radiation domination era

For the era of radiation domination, $p = \rho/3$, so the continuity equation (6) becomes

$$\dot{\rho} + 4\rho\frac{\dot{a}}{a} = 0, \tag{67}$$

with the solution of equation (67) is

$$\rho(t) = \rho_0 \frac{a_0^4}{a(t)^4}. \tag{68}$$

Note that, using the same reasoning as in the era of matter domination, then when equation (68) is entered to equation (1) and by doing some algebra steps, we can find

$$\dot{a}^2 + k = C_2^{\frac{1}{2-\beta}} a^{-\frac{2\beta}{2-\beta}}, \tag{69}$$

where

$$C_2 = 8\pi\, L_p^2\, \rho_0 \frac{a_0^4}{3}. \tag{70}$$

Equation (69) can be expressed in a simpler form, namely

$$\frac{da}{dt} = \sqrt{Ba^l - k}, \tag{71}$$

where $B = C_2^{\frac{1}{2-\beta}}$ and $l = -\frac{2\beta}{2-\beta}$. Note that, equations (71) and (13) are the same, only distinguished by constants A and B, and the definitions of m and l. That is, the form of the solution will also be the same and only be distinguished by the constants and definitions of m and l.

We can state general equation and general solution to the nonflat universe of these two eras in the following forms, respectively

$$\frac{da}{dt} = \sqrt{Ga^\eta - k} \tag{72}$$

and the solution

$$\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{\eta};\frac{1}{\eta}+1;\frac{Ga^\eta}{k}\right)}{\sqrt{Ga^\eta - k}} \sqrt{1 - \frac{Ga^\eta}{k}}\, a = t + K, \tag{73}$$

where $G = A$; $\eta = m$; $K = C$ for the era of matter domination, while $G = B$; $\eta = l$; and $K = C_3$ (although this integral constant is not shown here) for era of radiation domination.

Conclusion

As explained earlier, the main purpose of this paper is to determine the solution of the Friedmann modified equation as reviewed by Sheykhi in (Sheykhi, 2018), but for $k \neq 0$. In this work, a general solution was found in hypergeometric function for two eras, namely the matter domination era and the era of radiation domination. There have also been some special solutions especially for the era of matter domination with approximations of first and second order for $m = 0$ (or $\beta = 1/2$) and $m = 1$ (or $\beta = -1$). For $m = 0$ or $\beta = 1/2$ it can be seen that $\ddot{a} = 0$, which is clearly not in accordance with the fact that our universe is temporarily expanding. However, we might interpret this as a situation before or just before our universe underwent expanding.

Next we discuss the case of $\beta = -1$ for first order approximation. First, we review $\ddot{a}_1(t)$. To get $\ddot{a}_1 > 0$, we can do this by selecting $A < 0$ or $A > 0$. But physically, the most likely one for us to receive is $A < 0$ provided that $\rho_0 < 0$ or $a_0 < 0$. This means, that expanding really starts from the initial point/ state when ρ_0 or a_0 (one of them) is negative. Other conditions must also be met is $-\left(\frac{ik^2}{A^{\frac{3}{2}}} + t\right) > C$, where $t > 0$; of course k can be worth 1 or -1. While for $A > 0$, it must be t complex value. This certainly cannot be accepted physically. Second, we discuss $\ddot{a}_2(t)$. For $\ddot{a}_2(t)$ a positive value cannot be obtained, both for the selection of $A < 0$ and $A > 0$. Third, we discuss $\ddot{a}_3(t)$. For $\ddot{a}_3(t)$ similar to $\ddot{a}_1(t)$, that is, we can get $\ddot{a}_3(t) > 0$ for $A < 0$ and $A > 0$, but what is acceptable is $A < 0$ with the terms $-\frac{k^2 i}{A^{3/2}} + t > -C$ and $t > 0$. Whereas for $A > 0$, the reason is the same as $\ddot{a}_1(t)$ which is that

t is required to be complex, which is clearly difficult to interpret physically. Finally we see $\ddot{a}_4(t)$. For $\ddot{a}_4(t)$, we cannot get a positive value, which can only be obtained for $\ddot{a}_4(t) < 0$. For a second order approximation, specifically $\ddot{a}(t)$, of course we can choose certain conditions in the same way as in the case of first-order approximation to get the specific solutions we want, namely for $\ddot{a}(t) > 0$, but we don't do this here. The selection of conditions for the first order approximation is sufficient to represent the purpose of writing this paper.

The description that we describe is only limited to showing that the general solution we obtain can be used to find suitable specific solutions. The final section has also presented a general solution for two eras, namely the era of matter domination and the era of radiation domination. It can be seen that the detailed discussion is only carried out in the era of matter domination as an example; for the era of radiation domination, of course, it can also be easily calculated, but here we don't do it.

References

Alpher, R. A., & Herman, R. C. (1949). Remarks on the Evolution of the Expanding Universe. *Physical Review*, *75*(7), 1089.

Ambjørn, J., & Watabiki, Y. (2017). A modified Friedmann equation. *Modern Physics Letters A*, *32*(40), 1750224.

Amendola, L., & Tsujikawa, S. (2010). *Dark energy: theory and observations*. Cambridge University Press.

Barboza Jr, E. M., Nunes, R. da C., Abreu, E. M. C., & Neto, J. A. (2015). Dark energy models through nonextensive Tsallis' statistics. *Physica A: Statistical Mechanics and Its Applications*, *436*, 301–310.

Bertolami, O., & Páramos, J. (2014). Modified Friedmann equation from nonminimally coupled theories of gravity. *Physical Review D*, *89*(4), 44012.

Bonometto, S., Gorini, V., & Moschella, U. (2001). *Modern cosmology*. CRC Press.

Cheng, T.-P. (2009). *Relativity, gravitation and cosmology: a basic introduction* (Vol. 11). Oxford University Press.

Coquereaux, R. (2015). The history of the universe is an elliptic curve. *Classical and Quantum Gravity*, *32*(11), 115013.

Gamow, G. (1946). Expanding universe and the origin of elements. *Physical Review*, *70*(7–8), 572.

Ghaffari, S., Moradpour, H., Bezerra, V. B., Graça, J. P. M., & Lobo, I. P. (2019). Tsallis holographic dark energy in the brane cosmology. *Physics of the Dark Universe*, *23*, 100246.

Hidayat, T. (2010). Teori Relativitas Einstein: Sebuah Pengantar. *Bandung: Pe*.

Hikmawan, G., Soda, J., Suroso, A., & Zen, F. P. (2016). Comment on "Gauss-Bonnet inflation." *Physical Review D*, *93*(6), 68301.

Hubble, E. (1929). A relation between distance and radial velocity among extra-galactic nebulae. *Proceedings of the National Academy of Sciences*, *15*(3), 168–173.

Lymperis, A., & Saridakis, E. N. (2018). Modified cosmology through nonextensive horizon thermodynamics. *The European Physical Journal C*, *78*(12), 993.

Lyra, M. L., & Tsallis, C. (1998). Nonextensivity and multifractality in low-dimensional dissipative systems. *Physical Review Letters*, *80*(1), 53.

Pardede, K. F. S., Suroso, A., & Zen, F. P. (2018). Modified Friedmann equations in braneworld scenario. *Journal of Physics: Conference Series*, *988*(1), 12007. IOP Publishing.

Penzias, A. A., & Wilson, R. W. (1965). A measurement of excess antenna temperature at 4080 Mc/s. *The Astrophysical Journal*, *142*, 419–421.

Price, R. H. (2012). In an expanding universe, what does'nt expand? *American Journal of Physics*, *80*(376).

Ryder, L. (2009). *Introduction to general relativity*. Cambridge University Press.

Sheykhi, A. (2018). Modified Friedmann equations from Tsallis entropy. *Physics Letters B*, *785*, 118–126.

Soares, B. B., Barboza Jr, E. M., Abreu, E. M. C., & Neto, J. A. (2019). Non-Gaussian thermostatistical considerations upon the Saha equation. *Physica A: Statistical Mechanics and Its Applications*, 121590.

Tsallis, C., & Cirto, L. J. L. (2013). Black hole thermodynamical entropy. *The European Physical Journal C*, *73*(7), 2487.

Turok, N. (1983). The production of string loops in an expanding universe. *Physics Letters B*, *123*(6), 387–390.