International Journal of Innovation, Creativity and Change. www.ijicc.net
Special Edition: Safe Communities, 2020

The Innovation Iterative Method and its Stability in Time-Fractional Diffusion Equations

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In this research, we deal with the innovation or application iterative methods of an unconditionally implicit finite difference approximation equation and the one-dimensional, linear time fractional diffusion equations (TFDEs) via Caputo's time fractional derivative. Based on this implicit approximation equation, the corresponding linear system can be generated, in which its coefficient matrix is large scale and sparse. To speed up the convergence rate in solving the linear system iteratively, we construct the corresponding preconditioned linear system. Then we formulate and implement the Preconditioned Gauss-Seidel (PGS) iterative method for solving the generated linear system. Two examples of the problem are presented to illustrate the effectiveness of the PGS method. The two numerical results of this study show that the proposed iterative method is superior to the basic GS iterative method.

Keywords: *Caputo's fractional derivative, Implicit finite difference, PGS*

Introduction

Based on previous studies by Meerschaert & Tadjeran (2004); Sunarto & Sulaiman (2019); Sunarto, Sulaiman, & Saudi (2014); and Zhang (2009), many successful mathematical models based on fractional partial derivative equations (FPDEs) have been developed. There are several methods used to solve these models. For instance, we have the transform method from Çetinkaya & Kiymaz (2013); Chaves (1998); Gupta & Sharma (2010); Sene & Fall (2019), which is used to obtain analytical and/or numerical solutions of the fractional diffusion equations (FDEs). Other than this method, other researchers have proposed finite difference methods, such as explicit and implicit methods (Agrawal, 2002; Dey, 1999; Sun Cheng, 2003; Yuste & Acedo, 2005). Additionally, it is pointed out that the explicit methods are conditionally stable. Therefore, we discretise the time-fractional diffusion equation via the implicit finite difference discretisation scheme and Caputo's fractional partial derivative of order α in order to derive Caputo's implicit finite difference approximation equation. This approximation equation leads a tridiagonal linear system.

Due to the properties of the coefficient matrix of the linear system, which is sparse and large in scale, iterative methods are the alternative option for efficient solutions. As far as iterative methods are concerned, it can be observed that many researchers, such as Cheng, et al (2006); Hackbusch (2016); Saad (2003); and Young (1971), have proposed and discussed several families of iterative methods. In addition to that, the concept of block iteration has also been introduced by Evans (1985); Ibrahim & Abdullah (1995); Leblond, Rousselle, & Renaud (2003); Parter (1981); and Yousif & Evans (1986) to demonstrate the efficiency of its computation cost. Among the existing iterative methods, the preconditioned iterative methods have been widely accepted to be some of the efficient methods for solving linear systems (Bai, Huang, & Ng, 2007; Bo & Yang, 2012; Cheng et al., 2006; Gunawardena, Jain, & Snyder, 1991; Honghao,et al 2009; Ito & Toivanen, 2006; Rusten & Winther, 1992; Saad, 1996).

Because of the advantages of these iterative methods, the aim of this paper is to construct and investigate the effectiveness of the Preconditioned Gauss-Seidel (PGS) iterative method in solving time fractional parabolic partial differential equations (TPPDE's) based on Caputo's implicit finite difference approximation equation. To investigate the effectiveness of the PGS method, we also implement the Gauss Seidel (GS) iterative method being used a control method.

To demonstrate the effectiveness of the PGS method, let the time fractional parabolic partial differential equation (TPPDE's) be defined as

$$\frac{\partial^\alpha U(x,t)}{\partial^\alpha} = a(x)\frac{\partial^2 U(x,t)}{\partial x^2} + b(x)\frac{\partial U(x,t)}{\partial x} + c(x)U(x,t) \tag{1}$$

where *a(x), b(x),* and *c(x)* are known functions or constants and *α* is a parameter that refers to the fractional order of the time derivative.

The outline of this paper is organised as follows: In Sections 2 and 3, an approximate formula of Caputo's fractional derivative operator and the numerical procedure for solving the time fractional diffusion equation (1) by means of the implicit finite difference method are given. In Section 4, the formulation of the PGS iterative method is introduced. Section 5 shows a numerical example and its results and conclusions are given in Section 6.

Preliminaries

Before constructing Caputo's implicit finite difference approximation equation of Problem (1), the following are some basic definitions from fractional derivative theory used in this paper.

Definition 1: The Riemann-Liouville fractional integral operator J^{α} of order -α, according to Mei & Peng (2016) and Young (1971), is defined as:

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha} f(t) dt, \quad \alpha > 0 \; x > 0 \tag{2}$$

Definition 2: Caputo's fractional partial derivative operator D^{α} of order -α, according to Farid et al., (2019); Oliveira & Capelas De Oliveira (2019); and Young (1971), is defined as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0 \tag{3}$$

with $m-1 < \alpha \le m$, $m \in N$, $x > 0$

To obtain the numerical solution of Problem (1) with Dirichlet boundary conditions, firstly we derive an implicit finite difference approximation equation based on Caputo's derivative definition and the non-local fractional derivative operator. This implicit approximation equation can be categorised as an unconditionally stable scheme. To facilitate us in getting this approximation equation of Problem (1), let the solution domain of the problem be restricted to the finite space domain $0 \le x \le \gamma$, with $0 < \alpha < 1$, where the parameter α refers to the fractional order of the time derivative. In addition to that, consider the boundary conditions of Problem (1), given as

$U(0,t) = g_0(t)$, $U(\ell,t) = g_1(t)$, and the initial condition

$U(x,0) = f(x)$.

where $g_0(t)$, $g_1(t)$, and $f(x)$, are given functions. Discretise the approximation to the time fractional derivative in Eq. (1) by using Caputo's fractional partial derivative of order α, defined as (Baleanu, Wu, Bai, & Chen, 2017; Hackbusch, 2016; Sunarto, Sulaiaman, & Saudi, 2016; Young, 1971):

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-1)} \int_0^{\infty} \frac{\partial u(x-s)}{\partial t} (t-s)^{-\alpha} ds, \quad t > 0, \quad 0 < \alpha < 1 \tag{4}$$

Approximation for Fractional Diffusion Equation

According to Eq. (4), the formulation of Caputo's fractional partial derivative of the first order approximation method is given as

$$D_t^{\alpha} U_{i,n} \cong \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right) \tag{5}$$

and we have the following expressions:

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^{\alpha}}$$

and

$$\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}.$$

Before discretising Problem (1), let the solution domain of the problem be partitioned uniformly. To do this, we consider some positive integers: *m* and *n*, in which the grid sizes in space and time directions for the finite difference algorithm are defined as $h = \Delta x = \frac{\gamma - 0}{m}$ and $k = \Delta t = \frac{T}{n}$ respectively.

Based on these grid sizes, we construct the uniformly grid network of the solution domain, where the grid points in the space interval $[0, \gamma]$ are indicated as the numbers $x_i = ih,\ i = 0,1,2,...,m$ and the grid points in the time interval $[0, T]$ are labelled $t_j = jk,\ j = 0,1,2,...,n$. Then, the values of the function $U(x,t)$ at the grid points are denoted as $U_{i,j} = U(x_i, t_j)$.

By using Eq. (5) and the implicit finite difference discretisation scheme, Caputo's implicit finite difference approximation equation of Problem (1) at the grid point centred at $(x_i, t_j) = (ih, nk)$ is given as:

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right)$$

$$= a_1 \frac{1}{h^2} \left(U_{i-1,n} - 2U_{i,n} + U_{i+1,n} \right) + b_i \frac{1}{2h} \left(U_{i+1,n} - U_{i-1,n} \right) + c_i U_{i,n}, \tag{6}$$

for $i = 1,2...,m-1$.

Based on Eq. (6), this approximation equation is known as the fully implicit finite difference approximation equation, which is consistent regarding first order accuracy in time and second order in space. Basically, the approximation equation (6) can be rewritten based on the specified time level. For instance, we have for $n \geq 2$:

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega^{(\alpha)}\left(U_{i,n-j+1}-U_{i,n-j}\right)=\left(\frac{a_i}{h^2}-\frac{b_i}{2h}\right)U_{i-1,n}+\left(c_i-\frac{2a_i}{h^2}\right)U_{i,n}+\left(\frac{a_i}{h^2}+\frac{b_i}{2h}\right)U_{i+1,n}, \tag{7a}$$

$$\therefore \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}\left(U_{i,n-j+1}-U_{i,n-j}\right)=p_iU_{i-1,n}+q_iU_{i,n}+r_iU_{i+1,n},\ \text{where}$$

$$p_i=\frac{a_i}{h^2}-\frac{b_i}{2h},\ q_i=c_i-\frac{2a_i}{h^2},\ r_i=\frac{a_i}{h^2}+\frac{b_i}{2h}.$$

Also, for $n = 1$:

$$-p_iU_{i-1,1}+q_i^*U_{i,1}-r_iU_{i+1,1}=f_{i,1},\quad i=1,2,\ldots,m-1 \tag{7b}$$

where

$$\omega_j^{(\alpha)}=1,\ q_i^*=\sigma_{\alpha,k}-q_i,\ f_{i,1}=\sigma_{\alpha,k}U_{i,1}.$$

Based on Eq. (7b), it can be seen that the tridiagonal linear system can be constructed in matrix form as

$$A\underset{\sim}{U}=\underset{\sim}{f} \tag{8}$$

where

$$A=\begin{bmatrix} q_1^* & -r_1 & & & & \\ -p_2 & q_2^* & -r_2 & & & \\ & -p_3 & q_3^* & -r_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p_{m-2} & q_{m-2}^* & -r_{m-2} \\ & & & & -p_{m-1} & q_{m-1}^* \end{bmatrix}_{(m-1)x(m-1)}$$

$$\underset{\sim}{U}=\begin{bmatrix}U_{11} & U_{21} & U_{31} & \cdots & U_{m-2,1} & U_{m-1,1}\end{bmatrix}^T$$

$$\underset{\sim}{f}=\begin{bmatrix}U_{11}+p_1U_{01} & U_{21} & U_{31} & \cdots & U_{m-2,1} & U_{m-1,1}+p_{m-1}U_{m,1}\end{bmatrix}^T$$

Analysis of Stability

In this section, we have considered the stability analysis of the implicit finite difference approximation equation in Eq. (7). For stability analysis, we will use Von-Neumann's (Langlands & Henry, 2005; Zwillinger, 1992) and the Lax equivalence theorem (I., E., Richtmyer, & Morton, 1968; Jossey & Hirani, 2007; Schultz, 1966). It follows that the numerical solution of the approximation equation in Eq. (7) converges to the exact solution as $h,k\to 0$.

Theorem 1:

The fully implicit numerical method Eq.(7), the solution to Eq.(1) with $0 < \alpha < 1$ on the finite domain $0 \le x \le 1$, with zero boundary condition $U(0,t) = U(\ell,t) = 0$ for all $t \ge 0$, is consistent and unconditionally stable.

Proof:

To examine the stability of the proposed method, we find the solution of the form $U_j^n = \xi_n e^{i\omega jh}$, $i = \sqrt{-1}$, ω real. Therefore, Eq. (7) becomes

$$\sigma_{\alpha,k}\xi_{n-1}e^{i\omega jh} - \sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j+1}e^{i\omega jh} - \xi_{n-j}e^{i\omega jh}\right) =$$

$$- p_i\xi_n e^{i\omega(j-4)h} + \left(\sigma_{\alpha,k} - q_i\right)\xi_n e^{i\omega jh} - r_i\xi_n e^{i\omega(j+4)h} \tag{9}$$

By simplifying and reordering over Eq. (9), we have:

$$\sigma_{\alpha,k}\xi_{n-1} - \sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j+1} - \xi_{n-j}\right) = \xi_n\left(\left(\left(-p_i - r_i\right)\cos(\omega h)\right) + \left(\sigma_{\alpha,k} - q_i\right)\right).$$

this can be reduced to:

$$\xi_n = \frac{\xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j} - \xi_{n-j+1}\right)}{\left(1 + \frac{(p_i + r_i)}{\sigma_{\alpha,k}}\cos(\omega h) + \frac{q_i}{\sigma_{\alpha,k}}\right)} \tag{10}$$

In Eq. (10), it can be observed that

$$\left(1 + \frac{(-p_i - r_i)}{\sigma_{\alpha,k}}\cos(\omega h) - \frac{q_i}{\sigma_{\alpha,k}}\right) \ge 1,$$

for all α, n, ω, h and k we have:

$$\xi_1 \le \xi_0. \tag{11}$$

We also have

$$\xi_n \le \xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j} - \xi_{n-j+1}\right),\ n \ge 2. \tag{12}$$

Thus, for *n=2*, the last inequality implies

$$\xi_2 \le \xi_1 + \omega_2^{(\alpha)}\left(\xi_0 - \xi_1\right).$$

Again, repeating the above process, we can get

$$\xi_j \le \xi_{j-1}, \quad j=1,2,...n-1.$$

From Eq. (12), we finally have

$$\xi_n \le \xi_{n-1} + \sum_{j=2}^{n} \omega_j^{(\alpha)} \left(\xi_{n-j} - \xi_{n-j+1} \right) \le \xi_{n-j}.$$

Since each term in the summation is negative, it shows that inequalities in Eq. (11) and Eq. (12) imply

$$\xi_n \le \xi_{n-1} \le \xi_{n-2} \le ... \le \xi_1 \le \xi_0.$$

Thus,

$\xi_n = \left| U_j^n \right| \le \xi_0 = \left| U_j^0 \right| = \left| f_j \right|,$ which entails $\left\| U_j^n \right\| \le \left\| f_j \right\|,$ and we have stability.

Formulation of Preconditioned Gauss-Seidel

In relation to the tridiagonal linear system in Eq. (8), it is clear that the characteristics of its coefficient matrix are large-scale and sparse. As mentioned in Section 1, many researchers have discussed various iterative methods, such as (Bo & Yang, 2012; Cheng et al., 2006; Gunawardena et al., 1991; Hackbusch, 2016; Honghao et al., 2009; Saad, 2003; Young, 1971; Yousif & Evans, 1986). To obtain numerical solutions of the tridiagonal linear system (8), we consider the Preconditioned Gauss-Seidel (PGS) iterative method (Bo & Yang, 2012; Cheng et al., 2006; Gunawardena et al., 1991; Honghao et al., 2009), which is the most known and widely used for solving any linear system.

Before applying the PGS iterative method, we need to transform the original linear system (8) into the preconditioned linear system

$$A^* \underset{\sim}{x} = \underset{\sim}{f}^* \tag{13}$$

where

$$A^* = PAP^T,$$
$$\underset{\sim}{f}^* = P\underset{\sim}{f}, \quad \underset{\sim}{U} = P^T \underset{\sim}{x}.$$

The matrix P is called a preconditioned matrix, and is defined as (Gunawardena et al., 1991; Kohno, Kotakemori, Niki, & Usui, 1997)

where $P = I + S$ (14)

$$S = \begin{bmatrix} 0 & -r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -r_{m-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(m-1)x(m-1)}$$

and the matrix I is an identical matrix. To formulate the PGS method, let the coefficient matrix A^* in (8) be expressed as the summation of the three matrices

$$A^* = D - L - V \tag{15}$$

where *D*, *L,* and *V* are diagonal, lower triangular, and upper triangular matrices respectively.

By using Eq. (9) and (11), the formulation of PGS iterative method can be defined generally as (Bo & Yang, 2012; Cheng et al., 2006; Gunawardena et al., 1991; Honghao et al., 2009; Langlands & Henry, 2005)

$$\underset{\sim}{x}^{(k+1)} = (D-L)^{-1} V \underset{\sim}{x}^{(k)} + (D-L)^{-1} \underset{\sim}{f}^{*} \tag{16}$$

where $\underset{\sim}{x}^{(k+1)}$ represents an unknown vector at *(k+1)*th iterations. The implementation of the PGS iterative method can be described in Algorithm 1.

Algorithm 1: PGS

i. Initialise $\underset{\sim}{U} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.
ii. For $j = 1,2,\ldots,n$ Implement
   For $i = 1,2,\ldots,m-1$ calculate
   $\underset{\sim}{x}^{(k+1)} = (D-L)^{-1} V \underset{\sim}{x}^{(k)} + (D-L)^{-1} \underset{\sim}{f}^{*}$
   $\underset{\sim}{U}^{(k+1)} = P^T \underset{\sim}{x}^{(k+1)}$
   Convergence test. If the convergence criterion, i.e. $\left\| \underset{\sim}{U}^{(k+1)} - \underset{\sim}{U}^{(k)} \right\| \le \varepsilon = 10^{-10}$, is satisfied, go to Step (iii). Otherwise go back to Step (a).
iii Display approximate solutions.

Numerical Experiment

By using the approximation Eq. (7), we consider one example of the time fractional diffusion equation to test the effectiveness of the Gauss-Seidel (GS), and Preconditioned Gauss-Seidel (PGS) iterative methods. In order to compare the effectiveness of these two proposed iterative methods, three criteria have been considered: the number of iterations, execution time (in seconds), and maximum absolute error at three different values of α = 0.25, α = 0.50 and α = 0.75. For the implementation of both iterative schemes, the convergence test considered the tolerance error, which is fixed as $\varepsilon = 10^{-10}$.

Example 1 (Demir, Erman, Özgür, & Korkmaz, 2013):

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 < \alpha \le 1, 0 \le x \le \gamma, \quad t > 0 \tag{17}$$

where the boundary conditions are stated in fractional terms

$$U(0,t) = \frac{2kt^\alpha}{\Gamma(\alpha+1)}, U(\ell,t) = \ell^2 + \frac{2kt^\alpha}{\Gamma(\alpha+1)}, \tag{18}$$

and the initial condition

$$U(x,0) = x^2. \tag{19}$$

Following Eq. (17), as taking $\alpha = 1$, it can be seen that Eq. (17) can be reduced to the standard diffusion equation

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 \le x \le \gamma, \quad t > 0, \tag{20}$$

subjected to the initial condition

$$U(x,0) = x^2,$$

and boundary conditions

$$U(0,t) = 2kt, U(\ell,t) = \ell^2 + 2kt.$$

Then, the analytical solution of Eq. (19) is obtained as follows:

$$U(x,t) = x^2 + 2kt.$$

Now, by applying the series

$$U(x,t) = \sum_{n=0}^{m-1} \frac{\partial^n U(x,0)}{\partial t^n}\frac{t^n}{n!} + \sum_{n=1}^{\infty}\sum_{i=0}^{m-1} \frac{\partial^{mn+i} U(x,0)}{\partial t^{mn+i}} \frac{t^{n\alpha+i}}{\Gamma(n\alpha+i+1)}$$

To $U(x,t)$ for $0 < \alpha \le 1$, it can be shown that the analytical solution of Eq. (17) is given as

$$U(x,t) = x^2 + 2k\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Example 2 (Demir et al., 2013):

Let us consider the following time fractional initial boundary value problem, defined as

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \frac{1}{2}x^2 \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 < \alpha \le 1, 0 \le x \le \gamma, \quad t > 0 \tag{21}$$

where the boundary conditions are given in fractional terms

$$U(0,t) = 0, \quad U(1,t) = e^{t}, \tag{22a}$$

and the initial condition is

$$U(x,0) = x^2. \tag{22b}$$

Regarding Eq. (21) and $\alpha = 1$, it can be shown that Eq. (21) can also be reduced to the standard diffusion equation

$$\frac{\partial U(x,t)}{\partial t} = \frac{1}{2} x^2 \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 \le x \le \gamma, \quad t > 0 \tag{23}$$

Then, the analytical solution of Eq (23) is obtained as follows:

$$U(x,t) = x^2 e^{t}.$$

Now, by applying the series

$$U(x,t) = \sum_{n=0}^{m-1} \frac{\partial^n U(x,0)}{\partial t^n} \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sum_{i=0}^{m-1} \frac{\partial^{mn+i} U(x,0)}{\partial t^{mn+i}} \frac{t^{n\alpha+i}}{\Gamma(n\alpha+i+1)}$$

to $U(x,t)$ for $0 < \alpha \le 1$, it can be shown that the analytical solution of Eq (21) is stated as

$$U(x,t) = x^2 \left[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots \right]$$

All the results of numerical experiments for equations or problems (17) and (21), obtained from the implementation of GS and PGS iterative methods are recorded in Table 1 and Table 2. For different values of mesh sizes, m = 128, 256, 512, 1024, and 2048.

Conclusion

In order to get the numerical solution of time fractional diffusion problems, the paper presents the derivation of Caputo's implicit finite difference approximation equation in which this approximation equation leads a linear system. From the observation of all experimental results and by imposing the GS and PGS iterative methods, it is obvious at $\alpha = 0.25$ that the number of iterations has declined approximately by 64.87-99.82%. This corresponds to the PGS iterative method, compared with the GS method. Again, in terms of execution time, the implementations of PGS method are much faster, about 4.96-93.03% faster than the GS method. This means that the PGS method requires the least amount of iterations and computational time for $\alpha = 0.25$ when compared with GS iterative method. Based on the accuracy of both iterative methods, it can be concluded that their numerical solutions are in good agreement.

Figure 1: Graph Performance of GS and PGS Methods Example 1, Where α = 0.25, α = 0.50 and α = 0.75,
Mesh Size vs K

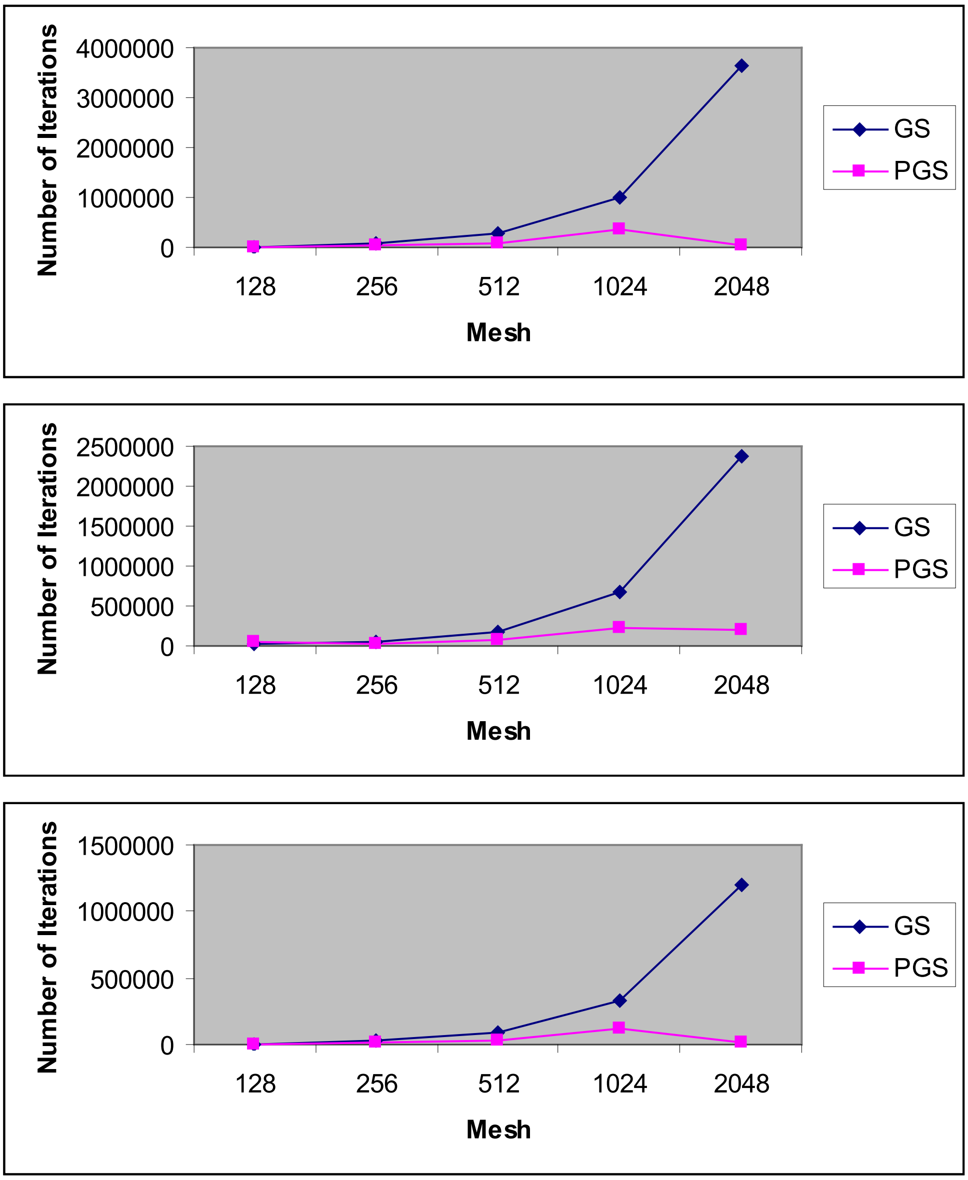


Figure 2: Graph Performance of GS and PGS Methods Example 1, where α = 0.25, α = 0.50 and α = 0.75,
Mesh Size vs Execution Time (Seconds)

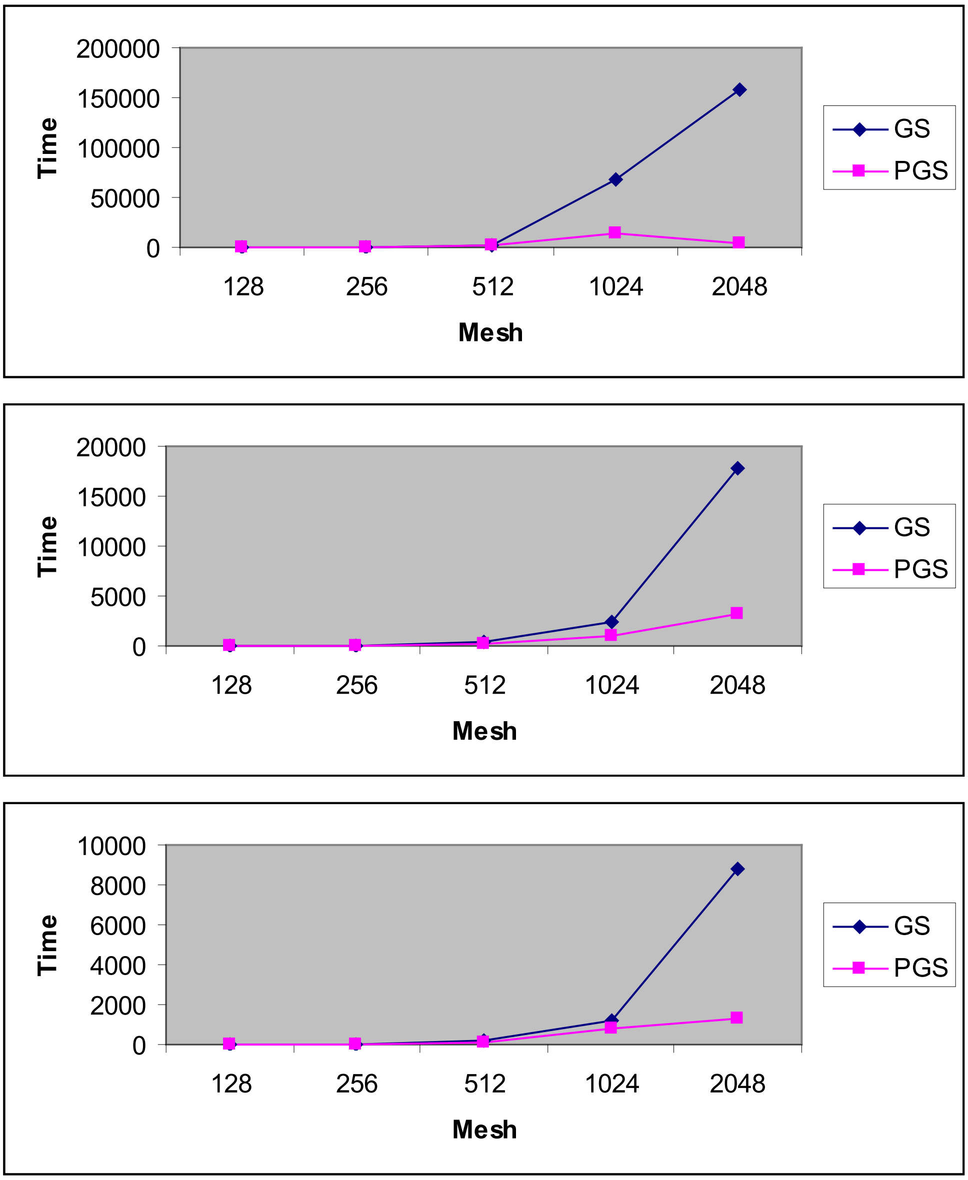
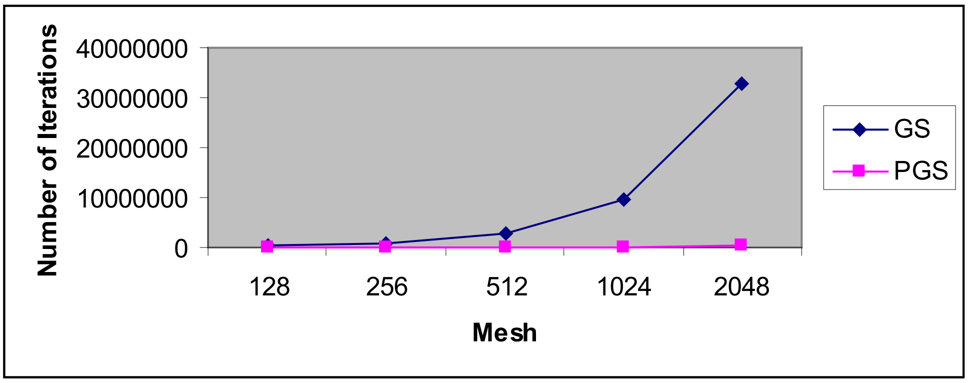
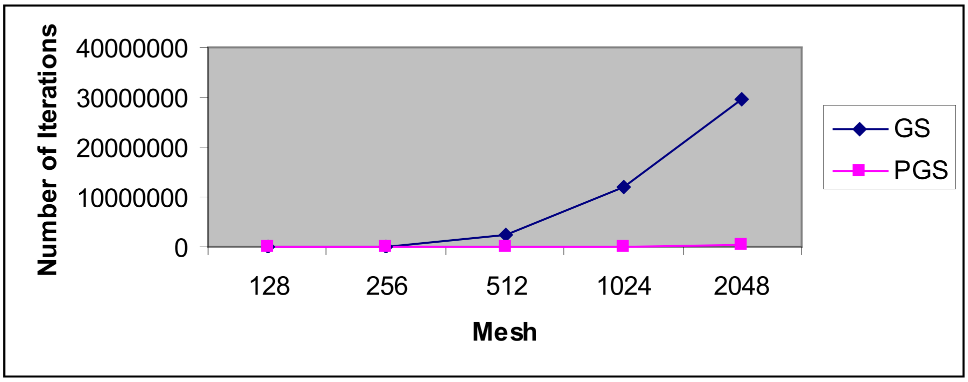


Figure 3: Graph Performance of GS and PGS Methods Example 2, where α = 0.25, α = 0.50 and α = 0.75,
Mesh Size vs K





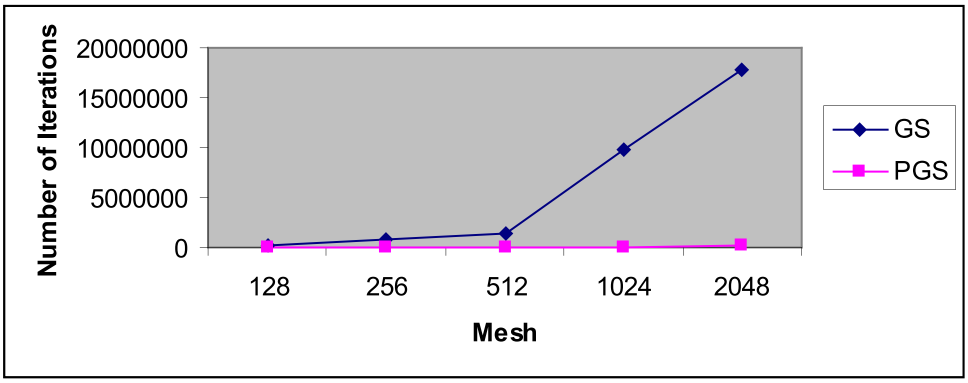


Figure 4: Graph Performance of GS and PGS Methods Example 2, where α = 0.25, α = 0.50 and α = 0.75,

Mesh Size vs Execution Time (Seconds)

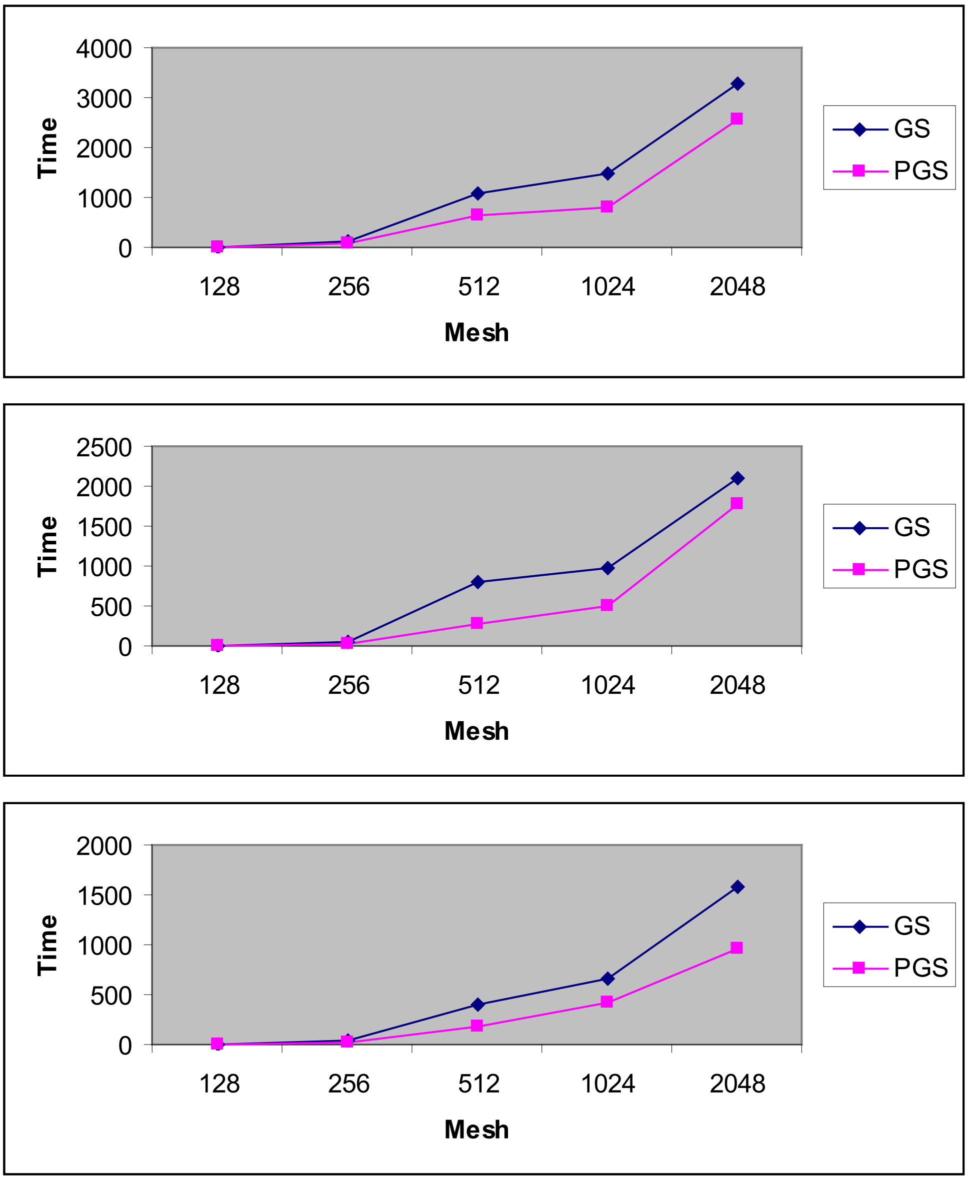


Table 1: Comparison of the Number Iterations (K), Execution Time (Seconds) and Maximum Errors for the Iterative Methods Using an Example where $\alpha = 0.25, 0.50, 0.75$

		α = 0.25			**α = 0.50**			**α = 0.75**		
M	**Method**	**K**	**Time**	**Max Error**	**K**	**Time**	**Max Error**	**K**	**Time**	**Max Error**
128	**GS**	21017	37.73	9.97e-05	13601	5.92	9.86e-05	6695	2.94	1.30e-04
	PGS	7292	35.86	9.96e-05	4715	2.23	9.84e-05	2319	1.93	1.30e-04
256	**GS**	77231	343.63	1.00e-04	50095	42.17	9.90e-05	24732	20.70	1.30e-04
	PGS	26884	261.56	9.98e-05	17417	16.68	9.87e-05	8585	12.37	1.30e-04
512	**GS**	281598	2747.34	1.02e-04	183181	339.85	1.01e-04	90783	166.75	1.32e0-4
	PGS	98422	1916.28	1.00e-04	63298	123.01	9.96e-05	31619	62.78	1.31e-04
1024	**GS**	1017140	68285.36	1.09e-04	663971	2454.53	1.08e-05	330622	1209.39	1.40e-04
	PGS	357258	14064.44	1.04e-04	232784	1007.47	1.03e-05	115617	820.93	1.35e-04
2048	**GS**	3631638	158914.30	1.38e-04	2380946	17795.25	1.38e-04	1192528	8794.26	1.71e-04
	PGS	21156	4104.17	1.36e-04	19153.0	3239.84	134e-05	12899	1305.5	1.35e-04

Table 2: Comparison of the Number Iterations (K), Execution Time (Seconds) and Maximum Errors for the Iterative Methods Using an Example where $\alpha = 0.25, 0.50, 0.75$

		α = 0.25			α = 0.50			α = 0.75		
M	**Method**	**K**	**Time**	**Max Error**	**K**	**Time**	**Max Error**	**K**	**Time**	**Max Error**
128	**GS**	230579	12.46	1.95e-02	182947	10.99	8.28e-02	112911	9.98	1.37e-01
	PGS	2873	8.48	1.95e-02	1398	7.00	8.28e-02	655	4.44	1.37e-01
256	**GS**	817596	110.24	1.95e-02	100946	53.98	8.29e-02	880921	35.98	1.30e-04
	PGS	10624	96.54	1.95e-02	5162	35.69	8.29e-02	2420	15.95	1.37e-01
512	**GS**	2853149	1071.25	1.95e-02	2282930	797.32	8.29e-02	1482921	397.32	1.37e-01
	PGS	39608	648.25	1.95e-02	18957	277.23	8.29e-02	8911	184.75	1.37e-01
1024	**GS**	9767783	1487.01	1.09e-02	11884877	964.92	8.29e-02	9884872	664.92	1.40e-04
	PGS	142635	791.55	1.95e-02	69108	492,97	8.29e-02	32602	420.11	1.37e-01
2048	**GS**	32773526	3266.51	1.38e-02	29754285	2106.87	8.29e-02	17752282	1585,23	1.37e-01
	PGS	487355	2543.23	1.95e-02	240051	1781.32	8.29e-02	116801	951,53	1.37e-01

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